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Generalized uncertainty relations and coherent and squeezed states

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Characteristic uncertainty relations and their related squeezed states are briefly reviewed and compared in accordance with the generalizations of three equivalent definitions of the canonical coherent states. The standard SU(1,1) coherent states are shown to be the unique states that minimize the Schrödinger uncertainty relation for every pair of the three generators and the Robertson relation for the three generators. The characteristic uncertainty inequalities are naturally extended to the case of several states. It is shown that these inequalities can be written in the equivalent complementary form.

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1. INTRODUCTION

The uncertainty principle is a basic feature of quantum physics. It was introduced by Heisenberg[1], who demonstrated an impossibility of the simultaneous precise measurement of the coordinate (q) and momentum (p) canonical observables by positing an approximate relation $\Delta p \Delta q \sim \hbar$, where $\hbar$ is the Planck constant. This relation was rigorously proved by Kennard[2] in the form of the inequality $(\Delta p)^2(\Delta q)^2 \geq \hbar^2/4$, where $(\Delta X)^2$ is the variance (dispersion) of the observable (Hermitian operator) X. Robertson[3] extended this inequality to arbitrary pair of operators X and Y,

$$(\Delta X)^2(\Delta Y)^2 \geq \frac{1}{4}\left|\langle[X,Y]\rangle\right|^2. \tag{1}$$

The Heisenberg–Kennard–Robertson inequality[1−3] (1) became known as the Heisenberg indeterminacy or uncertainty relation (UR) for X and Y, and here we shall follow this tradition.

According to inequality (1) the product of the uncertainties (precisions) ΔX and ΔY of the measurements of two quantum observables in one and the same state is not less than one half of the absolute mean value of the associated observable $C = -i[X,Y]$. Therefore this UR makes a statement about the preparation of a quantum state. However, by using the technique of positive operator-valued measure in measurement theory, one can extend UR (1) (with appropriately redefined notions of precisions ΔX and ΔY) to the joint measurement of two observables in the form of a slightly more stringent inequality[4], with one half instead of one fourth on the right-hand side of inequality (1).

The UR's are formal expressions of the uncertainty principle in quantum physics[1−3] and impose naturally fundamental limitations on the accuracy of measurements and telecommunications. This problem became of practical importance because of, e.g., experimental efforts to detect gravitational waves[5,6]. The main problem is how to optimize the intrinsic quantum fluctuations in the measurement process. Significant progress has been achieved in this direction in the last two decades by use the squeezed state technique[5,6]. The concepts of squeezed state[5−7] (SS) came from the observation that the equality in the Heisenberg UR for the canonical observables p and q can be maintained if the fluctuations of one of the two observables are reduced at the expense of the other. So the UR's play a dual role: They cause limitations on the measurement precision and in the same time indicate ways to improve the accuracy of the measurement devices. Thus a further study of the known UR's and their generalizations is of both theoretical and practical importance.

In this paper recent developments in the field of generalized SS's and UR's are considered and some new results are reported. The concept of SS's is closely related to that of coherent states[8,9] (CS's) introduced in 1963 patterned on the example of electromagnetic field oscillators in the pioneering works by Glauber, Klauder and Sudarshan (see Ref. 8, where a comprehensive list of references and reprints of selected articles is provided). The SS's for the canonical observables (the canonical SS's) are the unique one-mode states for which the three definitions of the canonical CS's (Refs. 8 and 9) are equivalently generalized, as is true in the multimode case as well[10].

The paper is organized as follows. The basic properties of the canonical CS's and canonical SS's are briefly reviewed in Section 2. A new inequality is pointed out, the minimization of which determines the canonical CS's uniquely. In Section 3 we consider the canonical SS's and show that they can be defined in three equivalent ways. The generalization of the SS to the case of arbitrary two observables on the basis of the more precise Schrödinger (or Schrödinger–Robertson) UR[11] is considered in section 4. SS's for several observables on the basis of Robertson UR for n observables are discussed in Section 5. The extension of the Schrödinger–Robertson UR's to all characteristic coefficients of the uncertainty matrix and to the case of two and several states is the subject of Section 6. Some applications of the ordinary and state-extended characteristic UR's are outlined; the main applications are the construction of observable induced metrics between quantum states and the finer classification of states, in particular of group–related CS's[8].

2. THE CANONICAL CANONICAL COHERENT STATES

The important overcomplete family $\{|\alpha\rangle\}$ of canonical CS's $|\alpha\rangle$, $\alpha \in \mathbf{C}$ (called also Glauber CS's), can be defined in three equivalent ways[8,9]:

(D1) As the set of eigenstates of boson destruction operator (the ladder operator) a: $a|\alpha\rangle = \alpha|\alpha\rangle$,

(D2) As the orbit through the ground state $|0\rangle$ ($a|0\rangle = 0$) constructed by use of the unitary displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$: $|\alpha\rangle = D(\alpha)|0\rangle$.

(D3) As the set of states which minimize the Heisenberg inequality (1) for the Hermitian components p and q of a with equal uncertainties $\Delta q = \Delta p$ ($a = (q+ip)/\sqrt{2}$; henceforth we work with dimensionless observables).

Let us note that in the definition (D3) one requires the minimization of inequality (1) for p, q plus the equality of the two variances. The set of states which minimize inequality (1) for p, q is much larger[9]. It is worth looking for another UR, the minimization of which determines the CS's $|\alpha\rangle$ uniquely. Such UR turned out to be the inequality

$$(\Delta q)^2 + (\Delta p)^2 \;\geq\; 1, \tag{2}$$

which follows from the obvious sequence $(\Delta q)^2 + (\Delta p)^2 \;\geq\; 2\Delta p \Delta q \;\geq\; 1$, and therefore is less precise than the Heisenberg inequality $\Delta p \Delta q \geq 1/2$.

The overcompleteness property reads ($\alpha = \alpha_1 + i\alpha_2$, $\mathrm{d}^2\alpha = \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$)

$$1 = \int |\alpha\rangle\langle\alpha| d\mu(\alpha), \quad \mathrm{d}\mu(\alpha) = \frac{1}{\pi}\mathrm{d}^2\alpha. \tag{3}$$

One may say that the family $\{|\alpha\rangle\}$ resolves the unity operator with respect to the measure $d\mu(\alpha)$ (overcompleteness of $\{|\alpha\rangle\}$ in the strong sense[8]). This relation provides the important analytic representation, known as canonical CS representation or the Fock-Bargmann analytic representation, in which $a = \mathrm{d}/\mathrm{d}\alpha$, $a^\dagger = \alpha$ and the state $|\Psi\rangle$ is represented by the function $\Psi(\alpha) = \exp(|\alpha|^2/2)\langle\alpha^*|\Psi\rangle$. In the years 1963-64 Klauder (see the references in Ref. 8) developed a general theory of the continuous representations and suggested the possibility of constructing overcomplete sets of states by use of irreducible representations of Lie groups.

There are at least three different ways (methods) of generalizing the canonical CS's that correspond to definitions (D1)–(D3) above[9]:

(D$'$1). The diagonalization of a non-Hermitian operator $L \neq L^\dagger$ (the eigenstate way, or the ladder operator method). The corresponding overcomplete (in the weak or strong sense[8]) families of states could be called L CS's or ladder operator CS's.

(D$'$2). The construction of orbit $\{|\vec{z}\rangle\}$ through a fixed vector $|\psi_0\rangle$ of a family of unitary operators $D(\vec{z})$ (orbit way or the displacement operator method). The corresponding CS can be called D CS's or displacement operator CS's.

(D$'$3). The minimization of appropriate UR $F[\psi] \geq 0$, where $F[\psi]$ is a functional of states $|\psi\rangle$ (the uncertainty way). The corresponding overcomplete families of states could be called U CS's or (optimal) uncertainty CS's.

The first two methods, especially the second of these, have received considerable attention and have been videly applied to various fields of physics[8,9], whereas the third one received significant attention only recently; see Refs. 12-25 and References therein. The developments of the second approach is thoroughly discussed in Refs. 8 and 9. Therefore in Section 3 I provide a brief review of the main steps in the first and the third ways only, noting main relationships between the three general definitions (D'). It appears that the (multimode) canonical SS's are the unique states, for which the three definitions $(D1) - -(D3)$ are equivalently generalized. It is worth noting that some authors (e.g., those of Ref. 9) were pessimistic about the possibility of effective and useful generalization of the third

defining property of canonical CS's to the case of more-complicated systems.

3. THE CANONICAL SQUEEZED STATES AS L, D, and U COHERENT STATES

Canonical CS $|\alpha\rangle$ diagonalizes the operator a, $[a, a^\dagger] = 1$, which is the ladder operator in the harmonic oscillator algebra $ho(1)$ spanned by $\{1, a, a^\dagger, a^\dagger a\}$. The subalgebra spanned by $\{1, a, a^\dagger\}$ is known as the Heisenberg–Weyl algebra, $h(1)$. This was the first and seminal example of diagonalization of a non-Hermitian operator. We stress that the eigenstates of a and of other non-Hermitian operators are not orthogonal; the term "diagonalization" is used for brevity and in analogy to the case of Hermitian operators. Chronologically the second example of L CS, to the best of the author's knowledge, was given in Refs. 26 and 27, where the diagonalization of the real and stationary[26] and complex and time-dependent[27] combinations of operators a, $a^\dagger$ has been performed ($\alpha \in \mathbf{C}$),

$$\begin{gathered} A(t)|\alpha;t\rangle = \alpha|\alpha;t\rangle, \\ A(t) = u(t)a + v(t)a^\dagger = A(u,v). \end{gathered} \tag{4}$$

The operator $A(t)$ was constructed[27] as a non-Hermitian invariant for the quantum varying frequency oscillator with Hamiltonian $H = (1/2)\left(p^2 + \omega^2(t)q^2/\omega_0^2\right)$, i.e., $A(t)$ had to obey the equation $\partial A/\partial t - i[A, H] = 0$. To satisfy this condition, the parameter $\varepsilon = (u-v)/\sqrt{\omega_0}$ was introduced and forced to obey the classical oscillator equation $\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0$. Here H is also dimensionless, and ω_0 is a frequency parameter that may be taken as $\omega(0)$. The dimensional Hamiltonian is $\hbar\omega_0 H$. The boson commutation relation $[A, A^\dagger] = 1$ was ensured by the Wronskian $\varepsilon^*\dot{\varepsilon} - \varepsilon\dot{\varepsilon}^* = 2i$. Then $\dot{\varepsilon} = i(u+v)\sqrt{\omega_0}$, $|u|^2 - |v|^2 = 1$, and $A(t) = U(t)A(0)U^\dagger(t)$, where $U(t)$ is the evolution operator and $A(0) = u_0 a + v_0 a^\dagger$.

In the coordinate representation the wave functions take the form of an exponential of a quadratic[27] (for the reader's convenience in this formula we restore the dimensions $x = q\sqrt{\hbar/m_0\omega_0}$, m_0 being a mass parameter)

$$\begin{aligned} \Psi_\alpha(x,t) = \langle x|\alpha;t\rangle = \frac{(\pi l_0^2)^{-1/4}}{(u-v)^{1/2}} \\ \times \exp\left[-\frac{1}{2l_0^2}\frac{v+u}{u-v}\left(x - \frac{\sqrt{2}\,l_0\alpha}{u+v}\right)^2\right] \\ \times \exp\left[\frac{1}{2}\left(\frac{u^*+v^*}{u+v}\alpha^2 - |\alpha|^2\right)\right], \end{aligned} \tag{5}$$

where $l_0 = (\hbar/m_0\omega_0)^{1/2}$ (a length parameter). Note that the time dependence is embedded completely in $u(t)$ and $v(t)$ [or, equivalently, in $\varepsilon(t)$ and $\dot{\varepsilon}(t)$] which justifies the notation $|\alpha;t\rangle = |\alpha, u(t), v(t)\rangle$. The wave functions [Eq. (5)] represent the time evolution of the canonical CS's $|\alpha\rangle$ if the initial conditions[27] $\varepsilon(0) = 1/\sqrt{\omega_0}$, $\dot{\varepsilon}(0) = i\sqrt{\omega_0}$ are imposed [then $u(0) = 1$, $v(0) = 0$]. Under these conditions $|\alpha, u(t), v(t)\rangle = U(t)|\alpha\rangle$. Time evolution of an initial $|\alpha, u_0, v_0\rangle$ for quadratic Hamiltonian system was studied in Ref. 7, where eigenstates of $ua + va^\dagger$ were denoted as $|\alpha\rangle_g$. The invariant $A(t)$ in Ref. 27 coincides with the boson operator $b(t)$ in Ref. 7. For different purposes invariants and wave functions for one-dimensional time-dependent quadratic systems were later studied in many papers[28,29]. Solutions to the Schrödinger equation for the nonstationary systems have been previously obtained, e.g., by Husimi and Chernikov[30], but with no reference to the eigenvalue problem and the invariants. Gaussian wave functions such as Eq. (5) have been studied by Schrödinger[31] and Kennard[2].

The nonstationary oscillator Hamiltonian is an element of the noncompact algebra $su(1,1)$ in the representation with Bargmann indices $k = 1/4, 3/4$, where the generators are ($K_\pm = K_1 \pm iK_2$)

$$K_3 = a^\dagger a/2 + 1/4, \quad K_- = a^2/2, \quad K_+ = a^{\dagger 2}/2.$$

Therefore $U(t) \in SU(1,1)$ and the set $\{|\alpha, u(t), v(t)\rangle\}$ is an $SU(1,1)$ orbit through the initial CS $|\alpha; u_0, v_0\rangle$. At $u_0 = 1$, $v_0 = 0$ this is an orbit through $|\alpha\rangle = D(\alpha)|0\rangle$,

$$|\alpha, u(t), v(t)\rangle = U(t)|\alpha\rangle = U(t)D(\alpha)|0\rangle. \tag{6}$$

The parameters u and v refer to the SU(1,1), and α refers to the Heisenberg–Weyl group H(1). The operator $U(t)D(\alpha)$ belongs to the semidirect product SU(1,1)∧H(1); therefore the set of states $|\alpha, u, v\rangle$ is an orbit of the group SU(1,1)∧H(1)[10,32]. This establishes the equivalence of the first two definitions (D′1) and (D′2) for the states $|\alpha, u, v\rangle$.

For non-quadratic Hamiltonians the invariant (the new boson annihilation operator) $A(t) = U(t)A(0)U^\dagger(t)$ is not linear in a and $a^\dagger$ and its eigenstates are no more of the form $|\alpha, u(t), v(t)\rangle$[33,10]. Therefore the term "coherent states for the nonstationary oscillator"[27] for $|\alpha;t\rangle = |\alpha, u, v\rangle$ is indeed adequate. The Hermitian components $P(t)$ and $Q(t)$ of $A(t)$ are also invariant, obey the canonical commutation relations $[Q(t), P(t)] = i$, and have the

physical meaning of coordinates of the initial point in the phase space[33]. Nonlinear realizations of boson operators $A_{(k)}$ (k-photon operators) are considered in the first paper of Ref. 35.

The orthonormalized eigenstates $|n,u,v\rangle$ of the quadratic invariant $A^\dagger(t)A(t)$ (an element of su(1,1)) were also constructed in Ref. 27. Note that any power of $A(t)$ and $A^\dagger(t)$ is again an invariant. In particular $A^\dagger(t)A(t)$ coincides with the Ermakov–Lewis invariant[36]. At the appropriate initial conditions the eigenstates of $A^\dagger(t)A(t)$ represent the time evolution of the Fock states $|n\rangle$.

For the s-dimensional quadratic system there are s linear in a_μ and $a_\mu^\dagger$ invariants $A_\mu(t) = u_{\mu\nu}(t)a_\nu + v_{\mu\nu}(t)a_\nu^\dagger \equiv A_\mu(u,v)$, which were simultaneously diagonalized[37],

$$A_\mu(u,v)|\vec{\alpha},u,v\rangle = \alpha_\mu|\vec{\alpha},u,v\rangle, \quad \mu = 1,\ldots,s. \tag{7}$$

The wave functions $\langle\vec{x}|\vec{\alpha},u,v\rangle$ are s dimensional Gaussians. This set of states is an orbit of the group Sp$(s,R)\wedge$ H(s) through the s dimensional vacuum state $|0\rangle$. In Section 4 we shall see that it can be considered family of U CS's related to the Robertson UR for $2s$ canonical observables[38]. Invariants and wave functions for nonstationary s-dimensional quadratic systems were later studied by many authors (see Ref. 28 and references therein).

By means of the known BCH formula for the transformation $S(\zeta)aS^\dagger(\zeta) = \cosh|\zeta|\,a - \sinh|\zeta|\exp(i\varphi)\,a^\dagger$, $\zeta = |\zeta|\exp(i\varphi)$, with the operator $S(\zeta) = \exp[(\zeta a^{\dagger 2} - \zeta^* a^2)/2] = \exp(\zeta K_+ - \zeta^* K_-)$ the solutions $|\alpha,u,v\rangle$ for the one-dimensional oscillator systems are immediately brought, up to a phase factor, to the form of famous Stoler states $|\zeta,\alpha\rangle = S(\zeta)|\alpha\rangle$ (Ref. 39) with cosh$|\zeta| = |u|$, $\varphi = \arg v - \arg u$,

$$|\alpha,u,v\rangle = \exp(i\arg u)\,\exp(\zeta K_+ - \zeta^* K_-)|\alpha\rangle. \tag{8}$$

Yuen[7] called the eigenstates $|\alpha,u,v\rangle$ of $ua + va^\dagger$ two photon CS's and suggested that the output radiation of an ideal monochromatic two photon laser is in a state $|\alpha,u,v\rangle$. Hollenhorst[5] named these states squeezed states to reflect that they exhibit fewer fluctuations in q or p than those in CS $|\alpha\rangle$. It is convenient to call these SS's canonical. They were intensively studied in quantum optics and are experimentally realized (see references in Ref. 40). The eigenstates $|n,u,v\rangle$ of $(ua+va^\dagger)^\dagger(ua+va^\dagger)$ became known as squeezed Fock states ($|n\!=\!0,u,v\rangle$ – squeezed vacuum) and the operator $S(\zeta)$ as a (canonical) squeeze operator[40,9]. Eigenstates $|\vec{\alpha},u,v\rangle$ [Eq. (7)] are known as multimode (canonical) SS's[41].

Radcliffe and Arecchi *et al*[42] introduced and studied the SU(2) analog of the states $|\alpha=0,u,v\rangle$ in the similar form to that of Stoler states [Eq (8)] i.e., spin CS's or atomic CS's. The results of Ref. 42 about the SU(2) CS's have been extended by Perelomov[43] to the noncompact group SU(1,1) and to any Lie group G as well; he succeeded in proving the Klauder suggestion for construction of overcomplete families of states by using unitary irreducible representations of Lie groups [group-related CS's (Ref. 8)]. For the discrete series $D^{(+)}(k)$ of SU(1,1), $k = 1/2, 1, \ldots$, and the lowest-weight reference vector these CS's [CS's with maximal symmetry or the standard SU(1,1) CS's] take a form similar to Eq. (8):

$$\begin{aligned}|\xi;k\rangle &= \exp(\zeta K_+ - \zeta^* K_-)|k,k\rangle \\ &= (1-|\xi|^2)^k\,\exp(\xi K_+)|k,k\rangle,\end{aligned} \tag{9}$$

where $|\xi| = \tanh|\zeta| \in \mathbf{D}_1$, $\arg\xi = -\arg\zeta + \pi$. The relation of these CS's to the U CS's, definition (D$'$3), was established later[15,22] on the basis of Schrödinger and Robertson UR's (see Section 4 below).

The third and seminal example of diagonalization of non-Hermitian operator was given in 1971 by Barut and Girardello[44], where they constructed the eigenstates of the SU(1,1) ladder operator K_- in the discrete series $D^{(\pm)}(k)$ and proved the overcompleteness of its eigenstates $|z;k\rangle$,

$$|z;k\rangle = N_{BG}\sum_{n=0}^\infty \frac{z^n}{(n!\Gamma(2k+n))^{1/2}}|k,k+n\rangle, \tag{10}$$

where $N_{BG} = [\Gamma(2k)/{}_0F_1(2k;|z|^2)]^{\frac{1}{2}}$, and ${}_0F_1(c;z)$ is the confluent hypergeometric function. The family of Barut–Girardello (BG) CS's $|z;k\rangle$ resolves the unity operator and provides a new analytic representation[44], which has been used in the diagonalization of more general su^c(1,1) operators[15,19,21–23]. This representation was recently extended to the boson realizations of the higher dimensional algebras $u(N,1)$[45] and $u(p,q)$[46].

For further developments in the direction of L-CS's, including the cases of Weyl ladder operators for the q-deformed h$_q$, su$_q$(2) and su$_q$(1,1), see, e.g., the brief review in Ref. 47 and references therein. The nonlinear CS's[48], which have enjoyed increasing interest recently[49], are also defined as eigenstates of non-Hermitian operators.

Every set of eigenstates $|z\rangle$ of a fixed non-Hermitian operator L, $L|z\rangle = z|z\rangle$, in particular the set of nonlinear CS's[48,49], the BG CS's $|z;k\rangle$ and their q–deformed extension, can be also defined according to (D$'$3) with equal variances of the Hermitian components X and Y of L, $L = X + iY$, on the basis of UR (1). The requirement of equal variances may be omitted if one finds a suitable less-precise inequality. It turned out that these same L CS can be uniquely defined as states which minimize the less-precise UR

$$(\Delta X)^2 + (\Delta Y)^2 \geq |\langle [X, Y]\rangle| \tag{11}$$

for the components of L. The proof of inequality (11) consists in the observation that $(\Delta X)^2 + (\Delta Y)^2 \geq 2\Delta X \Delta Y \geq |\langle [X, Y]\rangle|$. The minimization of inequality (11) occurs in the states with equal ΔX and ΔY only: $(\Delta X)^2 = (\Delta Y)^2 = |\langle [X, Y]\rangle|/2$. In general the representation of L CS's as D CS's is not possible, as proved for the family of BG CS's $|z;k\rangle$[23].

The larger family of canonical SS's $|\alpha, u, v\rangle$ can be uniquely determined[12,10] in the third equivalent way (as U CS) on the basis of the more-precise Schrödinger UR[11] for p and q,

$$(\Delta p)^2(\Delta q)^2 - (\Delta pq)^2 \ \geq\ 1/4, \tag{12}$$

where Δpq is the covariance of p and q, $\Delta pq = \langle pq + qp\rangle/2 - \langle p\rangle\langle q\rangle$. The three second moments of p and q in $|\alpha, u, v\rangle$ do not depend on α and read as[12] $(\Delta q)^2 = \frac{1}{2}|u - v|^2$, $(\Delta p)^2 = \frac{1}{2}|u + v|^2$, $\Delta pq = -\mathrm{Im}(uv^*)$. In other parameters they were calculated by Kennard[2], Stoler[39], in Ref. 10, and in Ref. 34. The above moments saturate inequality (12) identically with respect to u, v. One sees that the variance of p (q) tends to zero when $v \to -u$ ($v \to u$). Therefore these states can be called q–p ideal SS's.

By construction, the set $\{|\alpha, u, v\rangle\}$ is stable under the action of the evolution operator $U(t)$ of the varying frequency oscillator, $U(t)|\alpha, u_0, v_0\rangle = |\alpha, u(t), v(t)\rangle$. It was shown[10] that the most general Hamiltonian that keeps the canonical SS's stable is quadratic in p and q. If the time evolution is governed by a time-dependent quadratic Hamiltonian $H(t) = g_1(t)p^2 + g_2(t)(pq + qp) + g_3(t)q^2$ [where $g_i(t)$ are arbitrary differentiable functions] then an initial wave function of the form of Eq. (5), an initial SS, keeps this form for later times with some time-dependent $u(t)$ and $v(t)$. Here again the time dependence of the wave function $\Psi_\alpha(x, t)$ is completely embedded into parameters $u(t)$ and $v(t)$. $u(t)$ and $v(t)$ can be expressed in terms of $g_i(t)$ and a classical function $\varepsilon(t)$, which obeys the oscillator equation $\ddot{\varepsilon} + \Omega^2(t)\varepsilon = 0$ with "frequency"[37] $\Omega^2(t) = 4\omega_0^2 g_1 g_3 + 2\omega_0 g_2 \dot{g}_1/g_1 + \ddot{g}_1/2g_1 - 3\dot{g}_1^2/4g_1^2 - 4\omega_0^2 g_2^2 - 2\omega_0 \dot{g}_2$ and Wronskian $\varepsilon^*\dot{\varepsilon} - \varepsilon\dot{\varepsilon}^* = 2i$. Here $g_i(t)$ are dimensionless, $[\varepsilon] = [\omega_0]^{-1/2}$.

It is worth noting that the essential state parameters of SS $|\alpha, u, v\rangle$ (up to a phase factor) are four: In view of $u \neq 0$ one can rescale the parameters in Eq. (4) by dividing both sides by u. These parameters can be chosen in the form of two canonically conjugated pairs of classical observables[10]: $\langle p\rangle$, $\langle q\rangle$ and $\tilde{p} = \Delta pq/\Delta q$, $\tilde{q} = \Delta q$. For quadratic systems they satisfy the classical equations with Hamiltonian function $\mathcal{H} = \langle v(t), u(t), \alpha|H|\alpha, u(t), v(t)\rangle = \mathcal{H}(\langle p\rangle, \langle q\rangle, \tilde{p}, \tilde{q})$,

$$\frac{d\langle p\rangle}{dt} = -\frac{\partial\mathcal{H}}{\partial\langle q\rangle}, \qquad \frac{d\langle q\rangle}{dt} = \frac{\partial\mathcal{H}}{\partial\langle p\rangle}, \tag{13}$$

$$\frac{d\tilde{p}}{dt} = -\frac{\partial\mathcal{H}}{\partial\tilde{q}}, \qquad \frac{d\tilde{q}}{dt} = \frac{\partial\mathcal{H}}{\partial\tilde{p}}. \tag{14}$$

The stable evolution of quantum SS's is governed by these classical canonical equations. The time evolution of squeezing is controlled by the classical eqs. (14). If one restores the dimensions, one finds that in the limit $\hbar = 0$ the noisy variables $\tilde{p}$ and $\tilde{q}$ vanish, whereas the Eq. (13) recover the canonical classical equations with quadratic Hamiltonian function.

A sequence of different subsets of $\{|\alpha, u, v\rangle\}$ can be determined uniquely from the sequence of the UR's considered above: $[(\Delta p)^2 + (\Delta q)^2]^2/4 \geq (\Delta p)^2(\Delta q)^2 \geq (\Delta p)^2(\Delta q)^2 - (\Delta pq)^2 \geq 1/4$.

Thus the family of canonical SS's can be regarded equivalently as L CS's ($L = ua + va^\dagger$), D CS's ($D = \exp[(\zeta a^{\dagger 2} - \zeta a^2)/2]\exp(\alpha a^\dagger - \alpha^* a)$), and U CS's (Schrödinger U CS, Schrödinger optimal uncertainty states, correlated states[12] or Schrödinger intelligent states[15,19], the term "intelligent states" being introduced in Ref. 50).

4. SCHRÖDINGER INEQUALITY AND SQUEEZED STATES FOR TWO GENERAL OBSERVABLES

The concept of SS has been extended to noncanonical pair of observables, in particular to two generators of an arbitrary Lie group[13] on the basis of the equality in the Heisenberg UR: A set of SS's for two observables (Hermitian operators) X and Y was defined as the set of solutions to the eigenvalue equation $(X + i\lambda Y)|z, \lambda\rangle = z|z, \lambda\rangle$, where λ is real

parameter. Solutions to this equation for X and Y, the quadratures of a^2, were constructed in Ref. 14. A criterion was proposed[13] according to which a state $|\psi\rangle$ is squeezed if $(\Delta X)^2$ or $(\Delta Y)^2$ is less than $|\langle[X,Y]\rangle|/2$. This construction was generalized and refined in Ref. 15. The points are that the equality in inequality (1) is not invariant under the linear transformations of X and Y, in particular under the linear canonical transformations[17] and the inequality $(\Delta X)^2 \leq |\langle[X,Y]\rangle|/2$ can hold[15,22] for very large values of the fluctuation $(\Delta X)^2$. For example, the standard SU(1,1) CS $|\xi;k\rangle$ can exhibit strong squeezing according to the Eberly-Wodkiewicz criterion, whereas the fluctuations $(\Delta K_1)^2$ and $(\Delta K_2)^2$ are always grater than or equal to their value of $k/2$ in the ground state $|k,k\rangle$[15]. Besides, the Heisenberg UR for K_1 and K_2 is not minimized in every CS $|\xi;k\rangle$.

The appropreate UR to be used for the definition of SS's for two general observables X and Y is that of Schrödinger (or Schrödinger–Robertson)[11],

$$(\Delta X)^2(\Delta Y)^2 \geq \frac{1}{4}\,|\langle[X,Y]\rangle|^2 + (\Delta XY)^2, \quad (15)$$

where $\Delta XY \equiv \langle XY+YX\rangle/2 - \langle X\rangle\langle Y\rangle$ is the covariance of X and Y. It is more precise than inequality (1) and is reduced to that inequality when $\Delta XY = 0$. The set of X–Y SS's was defined[15] as the set of states that minimize inequality (15). Such minimizing states were called generalized intelligent states in Ref. 15. They could also be called X–Y correlated states[12] or (Schrödinger) optimal uncertainty states (optimal US). It was established that the X–Y SS's can be defined as solutions to the equations $(\lambda X + iY)|\psi\rangle = z|\psi\rangle$, where λ is complex parameter. To include the eigenstates of X, when they exist, one has to relax this condition slightly:

$$[u(X-iY)+v(X+iY)]\,|z,u,v\rangle = z|z,u,v\rangle, \quad (16)$$

$u,\ v,\ z \in \mathbf{C}$. The three second moments of X and Y in solutions $|z,u,v\rangle$ read as

$$\begin{aligned}(\Delta X)^2 &= \frac{1}{2}\frac{|u-v|^2}{|u|^2-|v|^2}\,i\langle[X,Y]\rangle,\\ \Delta XY &= \frac{\mathrm{Im}(u^*v)}{|u|^2-|v|^2}\,i\langle[X,Y]\rangle. \quad (17)\\ (\Delta Y)^2 &= \frac{1}{2}\frac{|u+v|^2}{|u|^2-|v|^2}\,i\langle[X,Y]\rangle.\end{aligned}$$

It turned out that the standard SU(1,1) and SU(2) CS's also minimize the Schrödinger inequality for the generators $K_1,\ K_2$ and $J_1,\ J_2$ respectively[15]. For example, the SU(1,1) CS's $|\xi;k\rangle$ are a particular case of the K_1–K_2 optimal US $|z,u,v;k\rangle$ corresponding to $z=-2k\sqrt{-uv}$ and $\xi=(-v/u)^{1/2}$.

The optimal US $|z,u,v\rangle$ can exhibit arbitrary strong squeezing of X and Y when the parameter v tends to $\pm u$[15]. Therefore the family of $|z,u,v\rangle$ is a family of X–Y ideal SS's. It is worth noting an important application of the K_i–K_j and J_i–J_j optimal US in the quantum interferometry: As shown by Brif and Mann the SU(1,1) and SU(2) intelligent states $|z,u,v\rangle$ which are not group-related CS's can greatly improve the sensitivity of the SU(2) and SU(1,1) interferometers[51]. Schemes for generation of SU(1,1) and SU(2) optimal US of radiation field can be found, e.g., in Refs. 23, 51, and 52.

From $(\Delta X)^2 \geq 0$ and Eqs. (17) it follows that if the commutator $i[X,Y]$ is positive (negative) definite then normalized eigenstates of $u(X-iY)+v(X+iY)$ exist for $|u|>|v|$ ($|u|<|v|$) only[22]. In such cases one can rescale the parameters and put $|u|^2-|v|^2=1$ ($|u|^2-|v|^2=-1$) as one normally does in the canonical case. It is also seen from Eqs. (17) that in the states with large $|\langle[X,Y]\rangle|$ the variances of both X and Y can be large. Thus the frequently used term "minimum uncertainty states" for states that minimize inequality (1) or (15) is in fact adequate in the case of canonical observables only: In general the lowest level, $\Delta_0 \leq \Delta X = \Delta Y$, can be reached in some subsets of $|z,u,v\rangle$. It then is natural for a given state to be considered squeezed if ΔX or ΔY is less than Δ_0[15,23].

The family of Schrödinger optimal US for the two quasi-spin operators K_1 and K_2 in the series $D^+(k)$ was first constructed, up to a normalization factor, in Ref. 15, and for the spin operators J_1 and J_2 in Ref. 50 (with no reference to Schrödinger inequality). For the quadratures of a^2 the states $|z,u,v\rangle$ were constructed in Ref. 20 and in fact Ref. 15. The even and odd CS's[53], which are the first examples of the intensively discussed macroscopic superpositions (see, e.g., Refs. 46, 49, and 54 and references therein), saturate inequality (15) with vanishing covariance because they are eigenstates of a^2. Minimization of inequality (15) for the quadratures of the q-deformed boson operator a_q is considered in Ref. 55, and for the quadratures of the q-deformed su(1,1) ladder operator $K_-(q)$ in Ref. 47.

Finally in this section, let us note that the minimization of the three UR's with increasing precision, inequalities (11), (1) and (15),

$$\frac{1}{4}[(\Delta X)^2+(\Delta Y)^2]^2 \;\geq\; (\Delta X)^2(\Delta Y)^2 \;\geq$$

$$(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2 \geq \frac{1}{4}\left|\langle [X,Y]\rangle\right|^2,$$

determines naturally a sequence of subsets of $\{|z,u,v\rangle\}$. One has

$$\{|z\rangle\} \subset \{|z,u,v\rangle|_{\mathrm{Im}uv^*=0}\} \subset \{|z,u,v\rangle\}, \quad (18)$$

where the smallest subset $\{|z\rangle\}$ consists of states $|z\rangle$ that minimize the less precise UR [inequality (11)]. These are solutions to any of the two eigenvalue equations $(X \pm iY)|z\rangle = z|z\rangle$, and, in the case of $X = q$, $Y = -p$ ($X = K_1$, $Y = K_2$), coincide with the Glauber CS's (BG CS). They are the eigenstates $|z\rangle$ that are the extension of the Glauber CS's to any two observables; The nontrivial example is given by the BG CS's $|z;k\rangle$. However the set of $|z\rangle$ is not always continuous. Nevertheless, this set of eigenvectors may still be "complete" in the finite Hilbert space, as one readily sees in the case of spin 1/2, for example.

5. ROBERTSON INEQUALITY AND SQUEEZED STATES FOR n OBSERVABLES

Two Hermitian operators (two observables) never close an algebra. Even in the simplest case of Heisenberg–Weyl algebra h(1) in fact the operators are three: p, q and 1 (the identity). Physical systems with higher symmetry are described by three and more independent observables. It was Robertson[38] who first realized that there must be an UR "for all observables under consideration", "for we cannot in general expect that the conditions necessary to insure minimum uncertainty in one pair will be consistent with those which insure the minimum in other pairs". The relation that Robertson found for n observables $X_1, X_2, \ldots X_n$ reads as

$$\det\sigma(\vec{X}) \geq \det C(\vec{X}), \quad (19)$$

where $\vec{X} \equiv (X_1, X_2, \ldots, X_n)$, $\sigma(\vec{X})$ is the $n\times n$ matrix of the second moments of observables, $\sigma_{ij} = \langle X_iX_j + X_jX_i\rangle/2 - \langle X_i\rangle\langle X_j\rangle \equiv \Delta X_iX_j$, $i,j = 1,2,\ldots,n$, and $C(\vec{X})$ is the $n\times n$ matrix of the first moments of the commutators $[X_i,X_j]$, $C_{kj} = -\frac{i}{2}\langle [X_k,X_j]\rangle$. For $n = 2$, inequality (19) coincides with (15). With minor changes the Robertson proof of inequality (19) is provided in the appendix of Ref. 47.

The Schrödinger UR proved to be efficient in defining the ideal SS for two observables[15], which encouraged us to define the SS's for several general observables as states that minimize Robertson UR (19)[23]. The latter definition is more effective for the even number of observables, $n = 2s$, because for odd number the right-hand side of UR (19) is vanishing. The minimization of UR (19) is considered in detail in Ref. 22, where the minimizing states are called Robertson intelligent states[22] or Robertson optimal US's[47]. For even-number $n = 2s$ the minimizing states are eigenstates of one real or s complex linear combinations of X_j, whereas for odd n the diagonalization of one real combination is necessary and sufficient.

For even $n = 2s$, keeping the analogy to the case of two observables, we define the s "ladder" operators $\tilde{a}_\mu = X_\mu + iX_{\mu+s}$ and their s complex combinations

$$\tilde{A}_\mu(u,v) := u_{\mu\nu}\tilde{a}_\nu + v_{\mu\nu}\tilde{a}^\dagger_\nu = \beta_{\mu j}X_j, \quad (20)$$

where $\beta_{\mu\nu} = u_{\mu\nu} + v_{\mu\nu}$, $\beta_{\mu,s+\nu} = i(u_{\mu\nu} - v_{\mu\nu})$. The condition for the equality in (19) is the simultaneous diagonalization of $\tilde{A}_\mu(u,v)$:

$$\tilde{A}_\mu(u,v)|\vec{z},u,v\rangle = z_\mu|\vec{z},u,v\rangle, \quad (21)$$

$\mu = 1,\ldots,s$. In the minimizing states $|\vec{z},u,v\rangle$ the second moments of X_μ can be expressed in terms of the first moments of their commutators:

$$\sigma = \mathcal{B}^{-1}\begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^{\mathrm{T}} & 0\end{pmatrix}\mathcal{B}^{-1\mathrm{T}}, \quad (22)$$

$$\mathcal{B} = \begin{pmatrix} u+v & i(u-v) \\ u^*+v^* & i(v^*-u^*)\end{pmatrix},$$

where $\tilde{C}_{\mu\nu} = \frac{1}{2}\langle[\tilde{A}_\mu, \tilde{A}^\dagger_\nu]\rangle$ and $\sigma = \sigma(\vec{X};z,u,v)$ is the dispersion matrix. Note that here u, v and $\tilde{C}$ are $s\times s$ matrices, β is an $s\times n$ matrix, and $\mathcal{B}$ and σ are $n\times n$, $n = 2s$. We suppose that $\mathcal{B}$ is not singular. For two observables, $n = 2$, we have $\beta_{11} = u+v$, $\beta_{12} = i(u-v)$, and $\tilde{C} = i(|u|^2 - |v|^2)[X_1,X_2]$, and Eqs. (22) recover Eqs. (17). From Eq. (21) and the equivalence $\Delta X(\psi) = 0 \quad \leftrightarrow \quad X|\psi\rangle = x|\psi\rangle$ it follows that the variance $\Delta X_i(\vec{z},u,v)$ will tend to zero when $\beta_{\mu,k} \to 0$ for every $k \neq i$ and at least for one μ. If this can be managed for every $i = 1,\ldots,n$ then the set of $|\vec{z},u,v\rangle$ is a set of ideal SS for n observables.

A. Example 1: The canonical observables

Let $X_\mu = q_\mu$, $X_{s+\mu} = p_\mu$, where p_μ, q_μ are s pairs of canonical observables, $[q_\mu,p_\nu] = i\delta_{\mu,\nu}$. In this case $\tilde{a}_\mu = a_\mu\sqrt{2}$ and $\tilde{A}_\mu = A_\mu(u,v)\sqrt{2}$, where $A_\mu(u,v)$ are the Bogolyubov transforms of boson creation-annihilation operators. Their common eigenstates $|\vec{\alpha},u,v\rangle$ [Eq. (7)], were constructed in Ref. 37 and studied in many papers[41] (but with no reference to

the Robertson relation). Up to phase factors they coincide with the multimode (canonical) SS's[41]. We note here that they are ideal SS's for all p_μ and q_μ. The proof is based on the fact that the uncertainty matrix $\sigma(\vec{Q})$, $\vec{Q}=(\vec{p},\vec{q})$ is nonnegative definite and can be diagonalized by means of linear canonical transformations[17,18]: $\vec{Q}\;\rightarrow\;\vec{Q}\,'=\Lambda\vec{Q}$. The total symplectic Λ preserves the equality in the Robertson relation[22]. The variances of the Λ-transformed operators in the old state are equal to that of the old operators in the new state, $|\psi'\rangle = U(\Lambda)|\psi\rangle$, where U is the unitary generator of the symplectic transformation. So the new state is also Robertson optimal US if the old one is and vice versa. However, $|\vec{\alpha},u,v\rangle = U(u,v)D(\vec{\alpha})|0\rangle$, where the operator $U(u,v)D(\vec{\alpha})$ belongs to the semidirect product group Sp$(s,R)\wedge$H(s). Thus the canonical multimode SS's are simultaneously L, D, and U CS.

Becuase of the canonical commutation relations, the general expression [Eqs. (22)] for the uncertainty matrix simplifies: In states $|\vec{\alpha},u,v\rangle$ the $s\times s$ matrix $\tilde{C}$ is a multiple of the identity, $\tilde{C}=1/2$. As a result $\sigma(\vec{Q})$ becomes symplectic itself. In other parameters this $\sigma(\vec{Q})$ was calculated by Ma and Rhodes[41].

The macroscopic superpositions $|\vec{\alpha}\rangle_\pm$ of multimode Glauber CS's $|\vec{\alpha}\rangle$ and $|-\vec{\alpha}\rangle$ [the even and odd multimode CS's (Refs. 56)] are eigenstates of $a_\mu a_\nu$. Therefore $|\vec{\alpha}\rangle_\pm$ minimize Robertson UR for the quadratures of $a_\mu a_\nu$. Note that $a_\mu a_\nu$ are mutually commuting Weyl ladder operators of the algebra sp(n,R). Therefore $|\vec{\alpha}\rangle_\pm$ are the BG-type CS's for $sp(s,R)$[46].

B. Example 2: The three generators K_i of SU$(1,1)$.

For odd number of observables the Robertson UR is minimized in the eigenstates of their real combinations only. In this case the more general set of eigenstates $|z,u,v,w;k\rangle$ of complex combination $uK_- + vK_+ + wK_3$ of K_i (the general operator of $su^c(1,1)$) was constructed[19,21]. In terms of the orthonormalized eigenstates $|k,k+n\rangle$ of K_3 the states with $u\neq 0$ read

$$|z,u,v,w;k\rangle = \mathcal{N}\sum_{n=0}^{\infty}\left(-\frac{l+w}{2u}\right)^n\sqrt{\frac{(2\kappa)_n}{n!}}\times{}_2F_1\left(\kappa+\frac{z}{l},-n;2\kappa;\frac{2l}{l+w}\right)|k,n+k\rangle, \quad (23)$$

where $\mathcal{N}$ is the normalization factor (the explicit form $\mathcal{N}(z,u,v,w,k)$ can be found in[23]), $l=\sqrt{w^2-4uv}$, $(a)_n$ is Pochhammer symbol and ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. The states are normalizable if at least one of the two inequalities $|w\pm\sqrt{w^2-4uv}|<2|u|$ holds. These states minimize Robertson UR (19) for the three operators K_i iff Im$w=0$, $v=u^*$. Among the minimizing states there are the standard $SU(1,1)$ CS's $|\xi;k\rangle$, eq. (9), as well. The latter correspond to $u=\cosh^2 r$, $v=\sinh^2 r\,\exp(2i\theta)$, $w=\sinh(2r)\,\exp(i\theta)$, where tanh$r=|\xi|$ and $\theta=\arg\xi+\pi$. Moreover, if one calculate all the first and the second moments of K_i in $|\xi;k\rangle$[57] one will find that they minimize Schrödinger UR (15) for every pair K_i-K_j, i.e. the standard $SU(1,1)$ CS's exhibit maximal uncertainty optimality. For the pair K_1-K_2 this property of $|\xi;k\rangle$ was discovered in[15]. Furthermore the CS's $|\xi;k\rangle$ are the unique states to exhibit this maximal uncertainty optimality for the three observables K_i. This unique property of $|\xi;k\rangle$ can be proved most easily if one consider the system of three eigenvalue equations (no summation over i, j)

$$(\beta_i K_i + \beta_j K_j)|\psi\rangle = z_{ij}|\psi\rangle, \quad i<j, \quad (24)$$

every one of which is necessary and sufficient $|\psi\rangle$ to minimize (15) for K_i, K_j. In the standard $SU(1,1)$ CS's representation[9] or in the BG analytic representation (24) is a system of ordinary differential equations. In the BG CS representation it is obeyed by the analytic function $\exp(c\eta)$ of η, $|c|\leq 1$, only, the latter corresponding to the CS's $|\xi;k\rangle$ (for details see the Appendix in[47]). Let us recall however that these group-related CS's can't exhibit squeezing in ΔK_1 and ΔK_2: $(\Delta K_i)^2(\xi)\geq k/2$, $i=1,2$.

Similarly one can prove the maximal uncertainty optimality of the standard $SU(2)$ group-related CS's. These results can be extended to semisimple Lie groups – the corresponding CS's with lowest/highest weight reference vector are unique to minimize the Robertson UR for all generators and for the quadratures of the Weyl ladder operators as well.

6. CHARACTERISTIC UNCERTAINTY RELATIONS AND THEIR STATE EXTENSIONS

From the matrix theory is known[58] that $\det M$ is an invariant (under similarity transformations) characteristic coefficient of a matrix M. For an $n\times n$ matrix there are n such invariant coefficients $C_r^{(n)}$, $r=1,2,\ldots,n$, defined by means of the secular equation

$$0=\det(M-\lambda)=\sum_{r=0}^{n}C_r^{(n)}(M)(-\lambda)^{n-r}. \quad (25)$$

The characteristic coefficients $C_r^{(n)}$ are equal[58] to the sum of all principal minors $\mathcal{M}(i_1,\ldots,i_r;M)$ of order r. One has $C_0^{(n)}=1$, $C_1^{(n)}=\mathrm{Tr}\,M=\sum m_{ii}$, and $C_n^{(n)}=\det M$. For $n=3$ we have, for example, three principle minors of order 2.

In these notations Robertson inequality (19) reads as $C_n^{(n)}\left(\sigma(\vec{X})\right)\,\geq\,C_n^{(n)}\left(C(\vec{X})\right)$. Inasmuch as the principal submatrices of order r of the dispersion matrix and of the mean commutator matrix are in fact again dispersion and mean commutator matrices for the r observables, the Robertson UR was extended[24] in an invariant manner to all characteristic coefficients in the form

$$C_r^{(n)}[\sigma(\vec{X})]\,\geq\,C_r^{(n)}[C(\vec{X})], \tag{26}$$

$r=1,2,\ldots,n$. These invariant relations were called characteristic uncertainty relations[24]. Robertson relation (19) is one of them, and can be called the nth-order characteristic inequality. Schrödinger UR (15) in the characteristic form reads as $C_2^{(2)}[\sigma(X,Y)]\,\geq\,C_2^{(2)}[C(X,Y)]$.

The minimization of the first-order inequality in expression (26), $\mathrm{Tr}\,\sigma(\vec{X})=\mathrm{Tr}\,C(\vec{X})$, can occur in the case of commuting operators only, as $\mathrm{Tr}\,C(\vec{X})\equiv 0$. An important example of minimization of the second-order inequality was pointed out in Ref. 24. The spin and quasi-spin CS's $|\tau;j\rangle$ and $|\xi;k\rangle$ minimize the second-order characteristic inequality for the three generators $J_{1,2,3}$ and $K_{1,2,3}$ respectively. From the results of Section 5 (see also Ref. 47) it follows that the standard SU(1,1) and SU(2) CS's are the unique states that minimize simultaneously the second- and the third-order characteristic inequalities for the corresponding three generators.

The characteristic inequalities relate combinations $C_r^{(n)}[\sigma(\vec{X};\rho)]$ of second moments of $X_1,\ldots,X_n$ in a (generally mixed) state ρ to the combinations $C_r^{(n)}[C(\vec{X};\rho)]$ of first moments of their commutators in the same state. However, there is no principal problem with which to compare the statistical properties of observables in different states. From the mathematical point of view the derivation of the state-extended characteristic UR's resorts on two simple matrix properties: (a) The sum of symmetric (antisymmetric) matrices is again a symmetric (antisymmetric) matrix and (b) the sum of nonnegative-definite matrices is again a nonnegative-definite matrix. The symmetricity of $\sigma(\vec{X})$, the antisymmetricity of $C(\vec{X})$, and the nonnegativity of $\sigma(\vec{X})$ and of $R(\vec{X})=\sigma(\vec{X})+iC(\vec{X})$ are the crucial properties, that the Robertson derivation of the inequality (19), and of (26) as well, relies on[38] (see also the proof in Ref. 47). Therefore we can rewrite the characteristic UR's for the sum of several uncertainty and mean commutator matrices that correspond to different, generally mixed, states ρ_m, $m=1,\ldots,m_s$,

$$C_r^{(n)}\left[\sum\nolimits_m\sigma(\vec{X};\rho_m)\right]\,\geq\,C_r^{(n)}\left[\sum\nolimits_m C(\vec{X};\rho_m)\right]. \tag{27}$$

These are extended characteristic uncertainty inequalities for n observables (extended to several states). For $r=n$ in inequality (27) we have the extension of the Robertson relation to the case of several states

$$\det\left[\sum\nolimits_m\sigma(\vec{X},\rho_m)\right]\,\geq\,\det\left[\sum\nolimits_m C(\vec{X},\rho_m)\right]. \tag{28}$$

Inasmuch as $\det\sum\sigma_m\,\neq\,\sum\det\sigma_m$ these are indeed new uncertainty inequalities, which extend the Robertson inequality to several states. We note that extended relations (27) and (28) are invariant under the nondegenerate linear transformations of the operators $X_1,\ldots,X_n$. If those operators span a Lie algebra, then we obtain the invariance of inequality (27) under the Lie group action in the algebra. If in pure states $|\psi_m\rangle$ inequality (28) is minimized, then it is minimized also in the states $U(g)|\psi_m\rangle$ as well, where $U(g)$ is the unitary representation of the group G. For two observables X and Y and two states $|\psi_{1,2}\rangle$ that minimize Schrödinger inequality (15), inequality (28) produces

$$\begin{aligned}\frac{1}{2}\left[\Delta XX(\psi_1)\Delta YY(\psi_2)+\Delta XX(\psi_2)\Delta YY(\psi_1)\right]&\\ -\Delta XY(\psi_1)\Delta XY(\psi_2)&\\ \geq\,\frac{1}{4}\langle\psi_1|[X,Y]|\psi_1\rangle\langle\psi_2|[Y,X]|\psi_2\rangle,&\end{aligned} \tag{29}$$

where, for the sake of symmetry, the variance $(\Delta X)^2$ of X in $|\psi\rangle$ was denoted as $\Delta XX(\psi)$. One can prove that this UR remains valid for any state[59]. It can be considered one of the basic UR's for quantum states. One can see that, if the two states coincide, inequality (29) recovers inequality (15). The minimization properties of the extended UR's remain to be considered elsewhere. Here we note that for $X=q$ and $Y=p$, inequality (29) is saturated by any two Fock states, Glauber CS's, or both, for example. The relation is also minimized in two equally squeezed states $|\alpha_1,u,v\rangle$ and $|\alpha_2,u,v\rangle$, $\mathrm{Im}(uv^*)=0$.

It is worth noting that, at $X=Y$, UR (29), and (15) as well, does not survive. This result encouraged me to look for an UR for two states and one observable. One solution is ($\langle i|X|j\rangle=\langle\psi_i|X|\psi_j\rangle$)

$$\left[(\Delta X(\psi_1))^2 + \langle 1|X|1\rangle^2\right]\left[(\Delta X(\psi_2))^2 + \langle 2|X|2\rangle^2\right] \geq \left|\langle 1|X^2|2\rangle\right|^2. \quad (30)$$

Both inequalities (15) and (30) follow from the Schwarz inequality, $|\langle\Psi_1|\Psi_2|^2 \leq ||\Psi_1||^2||\Psi_2||^2$, with suitably chosen $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Inequality (29) is different.

The extended UR can be used for construction of distances between quantum states. One simple new distance is based on UR (30), $D^2[\psi_1,\psi_2] = 2\,(1 - g(\psi_1,\psi_2;X))$, where

$$g(\psi_1,\psi_2;X) = \frac{\left|\langle\psi_2|X^2|\psi_1\rangle\right|}{\left(\langle\psi_1|X^2|\psi_1\rangle\langle\psi_2|X^2|\psi_2\rangle\right)^{1/2}}, \quad (31)$$

and X is any continuous or strictly positive observable. In this case $g(\psi_1,\psi_2;X) = 1$ if and only if $|\psi_1\rangle = |\psi_2\rangle$, and $D^2[\psi_1,\psi_2]$ does satisfy all the requirements for a distance. In particular one can take $X = 1$, which reproduces the well known Bures–Uhlmann distance (see the references in Ref. 60). In this way we establish the relation between the extended UR and the polarized distances[60].

Finally it is worth noting that every extended characteristic inequality can be written in terms of two new positive quantities, the sum of which is not greater than unity. To this end we put $C_r^{(n)}[\sigma(\vec{X},\rho)] = \alpha_r(1-P_r^2)$, where $0 \leq P_r^2 \leq 1$ (i.e., $1 - P_r^2 \leq 1$) and $\alpha_r \neq 0$. For $r = n$ one has (omitting index $r = n$) $\det\sigma(\vec{X},\rho) = \alpha\,(1-P^2)$. α_r may be viewed as scaling parameters. Then we can set $C_r^{(n)}[C(\vec{X},\rho)] = \alpha_r V_r^2$ and obtain from inequality (26) the inequalities for P_r and V_r, r=1,…,n,

$$P_r^2(\vec{X},\rho) + V_r^2(\vec{X},\rho) \leq 1. \quad (32)$$

The equality in expression (32) corresponds to the equality in expression (26) or (27). P_r, V_r are functionals of the state ρ [or of $\rho_1,\rho_2,\ldots$ in the case of extended inequalities (27)]. These can be called complementary quantities, and the form (32) of the extended characteristic relations can be called complementary form. Let us note that P_r and V_r are not uniquely defined by σ and C; they depend on the choice of the scaling parameter α_r. In the case of bounded operators X_i (say, spin components) the characteristic coefficients of σ and C are also bounded. Then α_r can be taken as the inverse maximal value of $C_r^{(n)}(\sigma)$. For one state and two observables with only two eigenvalues each, the complementary inequality (32) was recently considered in the important paper by Björk *et al*[25]. In this case the meaning of the complementary quantities P and V was elucidated to be that of the predictability (P) and the visibility (V) in the *welcher weg* experiment[25].

It is worth underlining that we have considered the developments of generalization of the SS's and UR's mainly along the lines of characteristic invariants of the uncertainty matrix. Schrödinger UR (12) is the simplest ordinary characteristic UR. Other types of ordinary UR's are also discussed in the literature[17,22,61–65]; UR for higher moments and universal invariants[62], trace-class UR's[17,22], parameter-based UR's[63], minimal-length UR's[64], etc. For an exhaustive list of references through 1986 see the review in Ref. 61.

7. CONCLUSION

The set of the characteristic uncertainty relations (UR's) and the related squeezed states (SS's) are briefly reviewed and compared in accordance with the generalizations of the three equivalent definitions of the canonical coherent states (CS's). It was shown that the multimode canonical SS's are the unique states (so far) for which the three definitions are equivalently generalized, where the basic uncertainty relation being that of Robertson (19). It was noted that the group-related CS's with the lowest (highest) weight reference vector minimize the Robertson relation for all generators and for the quadratures components of the Weyl ladder operators as well. The minimization of the other characteristic inequalities [inequality (26)] can be used for finer classification of group-related CS's. The standard SU(1,1) CS's were shown to be the unique states that minimize the third- and the second-order characteristic inequalities for the three generators. For two observables a new inequality, less precise than that of Heisenberg, is described that is minimized in Barut-Girardello-type CS's only.

It was proved that the characteristic uncertainty inequalities can be naturally extended to the case of several states. The state-extended uncertainty inequalities can be used for the construction of distances between quantum states. Further properties and applications of the new uncertainty relations remain to be considered elsewhere.

It was also shown here that the characteristic inequalities can be written in complementary form in terms of two positive quantities less than unity. In the case of one state and two observables with two eigenvalues each, the meaning of these complemen-

tary quantities were recently elucidated[25] to be that of the predictability and visibility in the *welcher weg* experiment.

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